UNIVERSITY OF LUXEMBOURG
ANALYSE 1A
2015-2016

EXERCISE SHEET 1

Remind

$$\begin{aligned}
\forall x, y, z \in \mathbb{R}: &\quad x \leq y \Rightarrow x + z \leq y + z \\
\forall x, y, z \in \mathbb{R}: &\quad [\ x \leq y \ \text{ and } \ 0 \leq z\ ] \Rightarrow xz \leq yz \\
\forall x, y \in \mathbb{R}: &\quad y - x = y + (-x) \\
\forall x \in \mathbb{R}: &\quad |x| = \max(x, -x)
\end{aligned}$$

1.1. **Verify the following identities.**

1.1.1. $[(x \leq y) \Rightarrow (0 \leq y - x)]$ and $[(x \leq y) \Rightarrow (x - y \leq 0)]$

1.1.2. $x \leq y \Leftrightarrow x + z \leq y + z$

1.1.3. $x \leq y \Leftrightarrow -y \leq -x$

1.1.4. $0 \leq x \Leftrightarrow -x \leq 0$

1.1.5. $[x \leq y \text{ and } z \leq t] \Rightarrow [x + z \leq y + t]$

1.1.6. $(-x)y = x(-y) = -(xy)$

1.1.7. $[x \leq y \text{ and } z \leq 0] \Rightarrow [zy \leq zx]$

1.1.8. $[x \geq 0 \text{ and } y \geq 0] \Rightarrow [xy \geq 0]$
What can we say about xy if $x \geq 0$ and $y \leq 0$, resp. if $x \leq 0$ and $y \leq 0$? Why?

1.1.9. $[0 \leq x \leq y \text{ and } 0 \leq z \leq t] \Rightarrow [xz \leq yt]$
Is the ipothesis "$x, y, z, t \geq 0$" necessary? Why?

Question: Which of the previous relations remain true in case of strict inequality?

1.2. **Choose the correct symbol of inequality and justify your answer.**

1.2.1. $x > 0 \Rightarrow \frac{1}{x} \dots 0$

1.2.2. $x < 0 \Rightarrow \frac{1}{x} \dots 0$

1.2.3. $x < y \Rightarrow \frac{1}{x} \dots \frac{1}{y}$

By consequences: $1 > \frac{1}{2} > \frac{1}{3} > \cdots > \frac{1}{n} > \frac{1}{n+1}$ si $n \in \mathbb{N}^*$, $n > 3$

1.3. **Answer the following questions:**

1.3.1. Let $x > 0$ and $y > 0$. Prove that $x \leq y \Leftrightarrow x^2 \leq y^2$ and $x < y \Leftrightarrow x^2 < y^2$. Is this still true if we don't assume x and y to be positive? Why?

1.3.2. Let $x > 0$. Compare x and $\frac{1}{x}$. Justify your answer.

1.3.3. **True or false?** $[x \leq y \text{ and } z \leq t] \Rightarrow [xz \leq yt] \quad \forall x, y, z, t \in \mathbb{R}$? Justify your answer.

1.3.4. **True or false?** $[x \leq y \text{ and } z \leq t] \Rightarrow [x - t \leq y - z]$? Justify your answer

1.4. Prove that $\forall\, a \in \mathbb{R}$ and $\varepsilon > 0$,

$$|x - a| \leq \varepsilon \quad \Leftrightarrow \quad a - \varepsilon \leq x \leq a + \varepsilon \ \Leftrightarrow \ x \in [a - \varepsilon, a + \varepsilon]$$
$$|x - a| < \varepsilon \quad \Leftrightarrow \quad a - \varepsilon < x < a + \varepsilon \ \Leftrightarrow \ x \in]a - \varepsilon, a + \varepsilon[$$

1.5. Let $b \in \mathbb{R}$ be a fixed real number, solve the inequality $x^2 \leq b^2$. Give a necessary and sufficient condition on a and b so that $a^2 \leq b^2$.

1.6. Let $b \in \mathbb{R}^*$ be a fixed real number, solve the inequality $\frac{1}{x} < \frac{1}{b}$.
Hint: consider different cases

1.7. Let $a, b \in \mathbb{R}_+^*$. Prove that

$$\frac{2ab}{a+b} \leq \sqrt{ab} \leq \frac{a+b}{2}$$

1.8. a) Let $a, b \in \mathbb{R}_+^*$. Prove that

$$\frac{a}{b} + \frac{b}{a} \geq 2$$

b) Let $a_1, a_2, \dots a_n \in \mathbb{R}_+^*$. Show that

$$(a_1 + a_2 + \cdots + a_n)(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}) \geq n^2$$

1.9. Let $0 < a < b$. Show that

$$0 < \frac{1}{b} < \frac{1}{a}$$

and

$$\frac{a}{1+a} < \frac{b}{1+b}.$$

Hint: $\frac{x}{1+x} = 1 - \frac{1}{1+x}$

1.10. Let A and B non empty, bounded subset in $\mathbb{R}$. Prove that $A \cup B$ has a supremum and $\sup(A \cup B) = \max\{\sup(A), \sup(B)\}$.

1.11. Calculate the supremum (if it exists) of the following sets (justify your answer!):
(1) $A = \{x \in \mathbb{Q} \mid x^3 < 7\}$;
(2) $B = \{x \in \mathbb{Q} \mid x^3 < 8\}$;
(3) $C = \{x \in \mathbb{R} \mid x = \frac{n}{n+1} \text{ for some } n \in \mathbb{N}\}$;
(4) $D = \{x \in \mathbb{R} \mid x = \frac{n^2}{n+1} \text{ for some } n \in \mathbb{N}\}$.

1.12. Let $A \subset \mathbb{R}$ be a lower bounded subset of $\mathbb{R}$. Denote with $\inf(A)$ the infimum of A, i.e. the greatest lower bound of A. Let $-A$ be the set obtained by changing the sign of each element of A. Prove that

$$\inf(A) = -\sup(-A)$$

and deduce that every non-empty, lower bounded subset of $\mathbb{R}$ has an infimum.